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On the Structure of the Newton–Padé Table

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In [1] the concept of minimal solution in the study of the Newton–Padé table was introduced, and its relationship with the Newton–Padé approximant was investigated. In this paper we show how we can use this concept to characterize the structure of the Newton–Padé table. Therefore we first study the table of minimal solutions. The results obtained exhibit some remarkable differences with the structure of the Padé table.

1. INTRODUCTION

Let $\{z_i\}_{i=0}^{\infty}$ be a sequence of (not necessarily distinct) points in the complex plane. Let $f(z)$ be a function which is holomorphic on some open set E containing these points, denoted by $f(z) \in H(E)$. If $f(z) \neq 0$ for $z \in E$, then we will use the notation $f(z) \in H^*(E)$. Then we can construct in a purely formal manner a Newton series associated with each $f(z) \in H(E)$,

$$f \equiv \sum_{i=0}^{\infty} f_{0i}\omega_{0i},$$

where we used the abbreviation $\omega_{00}(z) = 1$ and $\omega_{0i}(z) = (z - z_{i-1})\,\omega_{0,i-1}(z)$ for $i = 1, 2,\ldots$.

Let $(m, n) \in \mathbb{N}^2$ and $f(z) \in H(E)$; then the Newton–Padé approximation problem for $f(z)$ of order $[m, n]$ is defined as follows: Find two polynomials $p(z) = \sum_{i=0}^{m} a_{0i}\omega_{0i}(z)$ and $q(z) = \sum_{i=0}^{n} b_{0i}\omega_{0i}(z)$ satisfying:

$$\begin{aligned} &\text{(a)} \quad \partial p \leqslant m, \qquad \partial q \leqslant n, \\ &\text{(b)} \quad qf - p = \omega_{0,m+n+1}(z)\cdot v(z), \qquad \text{with} \quad v \in H(E). \end{aligned} \tag{1}$$

Here ∂ stands for "degree of."

Introducing the function σ: $H(E) \to \mathbb{N}$, defined by $\sigma(f) = n$ if and only if $f_{0i} = 0$ for $i = 0, 1,\ldots, n-1$ and $f_{0n} \neq 0$, then (b) is equivalent to $\sigma(qf - p) \geqslant m + n + 1$.

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The Newton–Padé problem (1) is equivalent to solving the homogeneous system of equations

$$\sum_{j=0}^{k} b_{0j} f_{jk} = a_{0k}, \qquad k = 0, 1,..., m \tag{2a}$$

$$= 0, \qquad k = m + 1,..., m + n \tag{2b}$$

for the unknowns $a_{00}, a_{01},..., a_{0m}$ and $b_{00}, b_{01},..., b_{0n}$. Hereby f_{jk} denotes the divided difference of order $k - j$ ($k \geqslant j$) determined for $f(z)$ in the points $z_i, z_{i+1},..., z_j$. By convention $f_{jk} = 0$ if $j > k$ and $b_{0j} = 0$ if $j > n$.

In [1] we introduced the concept of minimal solution by proving the following theorem.

THEOREM 1. *If rank* $[m, n] = n - d$, *then there exists a unique solution (up to a constant factor)* p_{mn}^*, q_{mn}^* *for* (1) *with* $\partial p_{mn}^* \leqslant m - d$ *and* $\partial q_{mn}^* \leqslant n - d$, *where at least one of the upper bounds is reached. Every other solution of* (1) *can be written in the form* $s(z) \cdot p_{mn}^*(z)$, $s(z) \cdot q_{mn}^*(z)$ *where* $s(z)$ *is a polynomial of degree less than or equal to* d.

In this theorem we denote by rank $[m, n]$, the rank of the coefficient matrix of the system (2b). By definition we call p_{mn}^*, q_{mn}^* the minimal solution of order $[m, n]$. The irreducible rational form $r_{mn} = p_{mn}/q_{mn}$, corresponding to p_{mn}^*/q_{mn}^*, is then called the Newton–Padé approximant of order $[m, n]$. Normalizing both, the minimal solution and the Newton–Padé approximant, such that q_{mn}^* and q_{mn} are monic, there exists the following relationship between them [1],

$$\begin{aligned} p_{mn}^*(z) &= d_{mn}(z) \cdot p_{mn}(z), \\ q_{mn}^*(z) &= d_{mn}(z) \cdot q_{mn}(z), \end{aligned} \tag{3}$$

with $d_{mn}(z) = \prod_{i=1}^{l} (z - z_{\alpha_i})$, $0 \leqslant l \leqslant \min(m, n)$ and with $\{z_{\alpha_i}\}_{i=1}^{l} \subset \{z_i\}_{i=0}^{m+n}$. Hereby we take as convention that $d_{mn}(z) = 1$ if $l = 0$. The points z_{α_i} appearing in $d_{mn}(z)$ are called *unattainable points* for $r_{mn}(z)$. We refer to [1] for a justification of this terminology.

Both the minimal solutions and the Newton–Padé approximants can be arranged in two-dimensional tables. These tables are called the *minimal solution table* and the *Newton–Padé table*, respectively. The tables are organized such that the element of order $[m, n]$ can be found on the intersection of the $m + 1$ column and the $n + 1$ row.

In the remaining part of this paper we will suppose that $f(z) \in H^*(E)$. This restriction is analogous to the restriction $c_0 \neq 0$ in the Padé approximation problem for $\sum_{i=0}^{\infty} c_i z^i$ (see, e.g., [2, p. 13]). It is necessary to prove properties for the complete tables, and to avoid limiting ourselves to their lower triangular parts.

In the next section we prove some theorems which give an insight into the structure of the table of minimal solutions. As an almost immediate consequence of these results we can derive some theorems which characterize the structure of the Newton–Padé table. This is done in Section 3. In the last section we give some examples, illustrating the theory.

The development of the theory illustrates the importance of the concept of minimal solution, which was first introduced in [1]. Indeed, the characterization of the Newton–Padé table can be derived almost immediately from the study of the table of minimal solutions.

2. Characterization Theorems for the Minimal Solution Table

In the following theorems we are interested in geometric figures of equal solutions. We will denote these figures by the indices of their corner elements.

THEOREM 2. *If* $\operatorname{rank}[m, n] = \operatorname{rank}[m-d, n-d] = n - d^*$, *where* $d^* \geqslant d$, *then all the minimal solutions lying in the triangle of the minimal solution table with corners* $[m-d, n-d]$, $[m-d, n+d]$, *and* $[m+d, n-d]$ *are equal to* $p^*_{m-d,n-d}$, $q^*_{m-d,n-d}$.

Proof. Since $\operatorname{rank}[m, n] = n - d^*$, we have that $\partial p^*_{m,n} \leqslant m - d^*$ and $\partial q^*_{m,n} \leqslant n - d^*$, where at least once the upper bound is reached. Moreover $\sigma(q^*_{mn}f - p^*_{mn}) \geqslant m + n + 1$. Consequently, using the definition of minimal solution,

$$p^*_{mn} = s(z) \cdot p^*_{m-d,n-d},$$

$$q^*_{mn} = s(z) \cdot q^*_{m-d,n-d},$$

where, in view of (3), $s(z)$ is a divisor of $\prod_{j=0}^{m+n} (z - z_j)$. However, since $\operatorname{rank}[m-d, n-d] = n - d^*$ we have $\partial p^*_{m-d,n-d} \leqslant m - d^*$ and $\partial q^*_{m-d,n-d} \leqslant n - d^*$, where at least one of the upper bounds is reached. Consequently $s(z) \equiv 1$. And hence $\sigma(q^*_{m-d,n-d}f - p^*_{m-d,n-d}) \geqslant m + n + 1$, which proves the theorem. ∎

If $\operatorname{rank}[m-d, n-d] < \operatorname{rank}[m, n] \leqslant n - d$ then $s(z)$, as defined in the proof of Theorem 2, does not need to be identically 1. Examples can be constructed where this phenomenon arises.

EXAMPLE 1. Let $z_{0+3i} = -3$, $z_{1+3i} = 0$, $z_{2+3i} = 1$, $z_{3+3i} = 2$ for $i = 0, 1, 2$, and let $f(-3) = \frac{1}{2}$, $f(0) = 2$, $f(1) = \frac{3}{2}$, $f(2) = \frac{4}{3}$, $f'(-3) = -\frac{1}{4}$, $f'(0) = -1$, $f'(1) = -\frac{1}{4}$, $f'(2) = 1$, $f''(-3) = -\frac{1}{4}$, $f''(0) = 1$, $f''(1) = \frac{1}{4}$, $f''(2) = 1$.

Then

$$p_{34}^* = (z+2)(z-2) \qquad \text{with rank } [3, 4] = 3$$
$$q_{34}^* = (z+1)(z-2)$$

and

$$p_{45}^* = (z+2)(z-2)\, z \qquad \text{with rank } [4, 5] = 4.$$
$$q_{45}^* = (z+1)(z-2)\, z$$

Hence we have a case as described in the remark preceding this example and we see that $s(z) = z \not\equiv 1$.

Since in Theorem 2, $\min(m - d^* - \partial p_{m-d,n-d}^*, n - d^* - \partial q_{m-d,n-d}^*) = 0$, there remains the possibility that $\max(m - d^* - \partial p_{m-d,n-d}^*, n - d^* - \partial q_{m-d,n-d}^*) > 0$. This possibility is treated in the next two corollaries.

COROLLARY 1. *If under the conditions of the theorem* $\partial q_{m-d,n-d}^* = n - d_2 - d^*$ *with* $d_2 > 0$ *and* $\text{rank}[m - d, n - d_2 - d] = n - d_2 - d^*$, *then all the minimal solutions lying in the triangle of the minimal solution table with corners* $[m - d, n - d_2 - d]$, $[m - d, n + d]$ *and* $[m + d_2 + d, n - d_2 - d]$ *(see Fig. 1) are equal to* $p_{m-d,n-d_2-d}^*$, $q_{m-d,n-d_2-d}^*$.

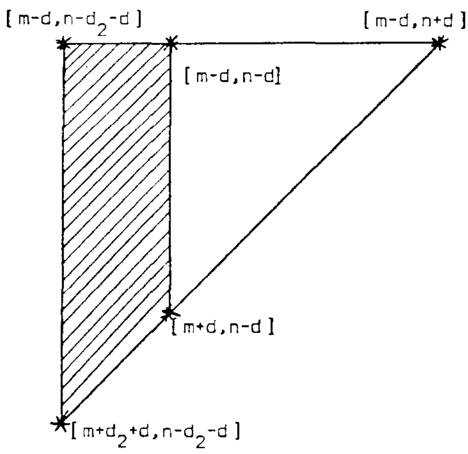

FIGURE 1

Proof. Since $\text{rank}[m - d, n - d_2 - d] = n - d_2 - d^*$ we necessarily have

$$p_{m-d,n-d}^* \equiv p_{m-d,n-d_2-d}^*,$$
$$q_{m-d,n-d}^* \equiv q_{m-d,n-d_2-d}^*,$$

which implies $\sigma(q_{m-d,n-d_2-d}^* f - p_{m-d,n-d_2-d}^*) \geqslant m + n + 1$. ∎

The condition $\operatorname{rank}[m-d, n-d_2-d] = n-d_2-d^*$ is necessary, since otherwise counterexamples can be constructed.

Analogously we have:

COROLLARY 2. *If under the conditions of the theorem* $\partial p^*_{m-d,n-d} = m-d_1-d^*$ *with* $d_1 > 0$ *and* $\operatorname{rank}[m-d_1-d, n-d] = n-d^*$, *then all the minimal solutions lying in the triangle of the minimal solution table with corners* $[m-d_1-d, n-d]$, $[m-d_1-d, n+d_1+d]$, *and* $[m+d, n-d]$ *(see Fig. 2) are equal to* $p^*_{m-d_1-d,n-d}$, $q^*_{m-d_1-d,n-d}$.

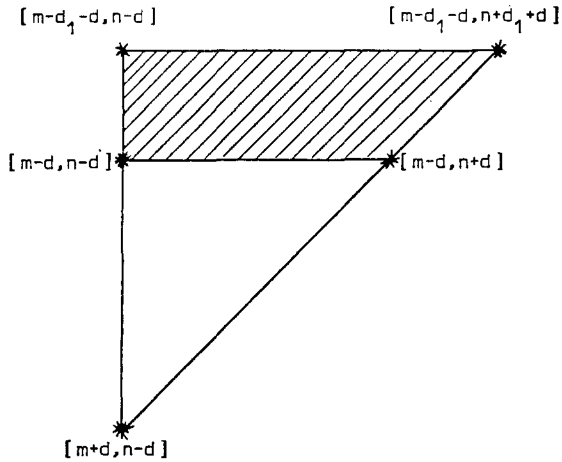

FIGURE 2

Hence if one of the degrees of the polynomials $p^*_{m-d,n-d}$, $q^*_{m-d,n-d}$, occurring in Theorem 2, does not reach its upper bound, this means that the triangle of equal minimal solutions possibly can be extended, either to the left or above, depending on whether it is $q^*_{m-d,n-d}$ or $p^*_{m-d,n-d}$ which does not reach its upper bound.

On the other hand, the position of the hypotenuse of the right-angled triangle is determined by the number $\sigma(q^*_{m-d,n-d}f - p^*_{m-d,n-d})$, see Fig. 3.

If $\sigma(q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2}) = m+n+1$ and $\partial p^*_{m-d_1,n-d_2} = m-d_1$, $\partial q^*_{m-d_1,n-d_2} = n-d_2$, then we say that $[m-d_1, n-d_2]$, $[m-d_1, n+d_1]$ and $[m+d_2, n-d_2]$ determine a *maximal triangle of equal minial solutions.* This terminology is made clear by the following theorem.

THEOREM 3. *If* $\partial p^*_{m-d_1,n-d_2} = m-d_1$, $\partial q^*_{m-d_1,n-d_2} = n-d_2$ *and* $\sigma(q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2}) = m+n+1$, *then* $p^*_{kl} = p^*_{m-d_1,n-d_2}$, $q^*_{kl} = q^*_{m-d_1,n-d_2}$ *if and only if* $[k, l]$ *belongs to the triangle with corners* $[m-d_1, n-d_2]$, $[m-d_1, n+d_1]$, *and* $[m+d_2, n-d_2]$.

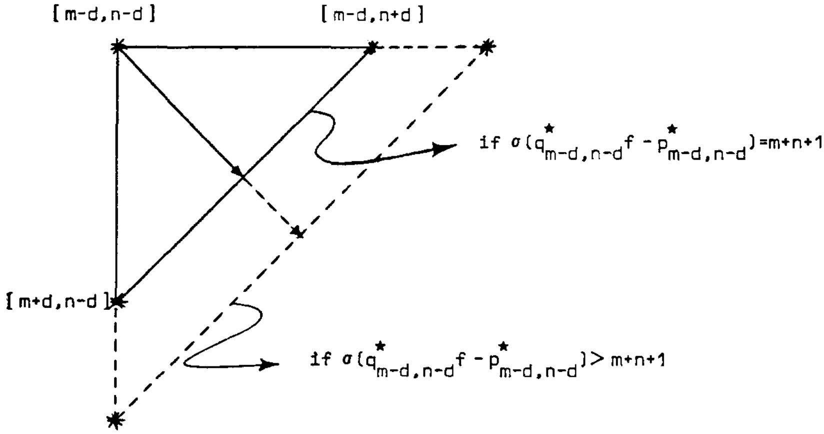

FIGURE 3

Proof. We only have to prove the necessary condition, since the sufficiency is obvious. Since $\partial p^*_{m-d_1,n-d_2} = m - d_1$ and $\partial q^*_{m-d_1,n-d_2} = n - d_2$, we necessarily have $k = m - d_1 + i$ and $l = n - d_2 + j$ with $i, j \geqslant 0$.

On the other hand,

$$m + n + 1 = \sigma(q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2}) = \sigma(q^*_{kl}f - p^*_{kl})$$

implies that $k + l + 1 \leqslant m + n + 1$, or $i + j \leqslant d_1 + d_2$. ▮

This theorem can also be formulated in a slightly more general way.

THEOREM 4. *If* $\partial p^*_{m-d_1+i,n-d_2+j} = m - d_1$, $\partial q^*_{m-d_1+i,n-d_2+j} = n - d_2$, *and* $\sigma(q^*_{m-d_1+i,n-d_2+j}f - p^*_{m-d_1+i,n-d_2+j}) = m + n + 1$, *where* $i, j \geqslant 0$ *and* $i + j \leqslant d_1 + d_2$, *and if* $\operatorname{rank}[m - d_1, n - d_2] = n - d_2$, *then* $p^*_{kl} \equiv p^*_{m-d_1,n-d_2}$, $q^*_{kl} \equiv q^*_{m-d_1,n-d_2}$ *if and only if* $[k, l]$ *belongs to the triangle with corners* $[m - d_1, n - d_2]$, $[m - d_1, n + d_1]$ *and* $[m + d_2, n - d_2]$.

Proof. In view of the initial hypothesis,

$$p^*_{m-d_1+i,n-d_2+j} = s(z) \cdot p^*_{m-d_1,n-d_2},$$

$$q^*_{m-d_1+i,n-d_2+j} = s(z) \cdot q^*_{m-d_1,n-d_2}.$$

However, since $\operatorname{rank}[m - d_1, n - d_2] = n - d_2$, $s(z) \equiv 1$. Consequently, we have $\partial p^*_{m-d_1,n-d_2} = m - d_1$ and $\partial q^*_{m-d_1,n-d_2} = n - d_2$. Hence the previous theorem can be applied. ▮

This theorem no longer holds if $\operatorname{rank}[m - d_1, n - d_2] < n - d_2$. However, then there exists a "parallelogram" of equal minimal solutions. This is shown by the next theorem.

THEOREM 5. *If* $\partial p^*_{m-d_1,n-d_2} = m - d_1 - d_1^*$, $\partial q^*_{m-d_1,n-d_2} = n - d_2 - d_2^*$ *and* $\sigma(q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2}) = m + n + 1$ *with* $d_1^*, d_2^* \geq 0$ *and* $\max(d_1^*, d_2^*) > 0$, *and if* $\text{rank}[m - d_1 - 1, n - d_2] \neq \text{rank}[m - d_1, n - d_2]$, *then* $p^*_{kl} = p^*_{m-d_1,n-d_2}$, $q^*_{kl} = q^*_{m-d_1,n-d_2}$ *if and only if* $[k, l]$ *belongs to the "parallelogram" with corners* $[m - d_1 + d_2^*, n - d_2 - d_2^*]$, $[m + d_2 - d_2^*, n - d_2 - d_2^*]$, $[m - d_2 - d_1^*, n - d_2 + d_1^*]$, *and* $[m - d_1 - d_1^*, n - d_2 + d_1^*]$.

Proof. Again we confine ourselves to prove the necessary condition, since the sufficiency is rather obvious. Analogously as in Theorem 3 one proves that $k = m - d_1 - d_1^* + i$, $l = n - d_2 - d_2^* + j$ with $i, j \geqslant 0$ and $i + j \leqslant d_1 + d_2 + d_1^* + d_2^*$. To prove the theorem it then suffices to show that $i + j \geqslant d_1^* + d_2^*$. This will be the case if we prove that $\partial p^*_{kl} < m - d_1 - d_1^*$ and $\partial q^*_{kl} < n - d_2 - d_2^*$ as soon as $[k, l]$ belongs to the triangle with corners $[m - d_1 - d_1^*, n - d_2 - d_2^*]$, $[m - d_1 + d_2^* - 1, n - d_2 - d_2^*]$, and $[m - d_1 - d_1^*, n - d_2 + d_1^* - 1]$. Therefore we note that since $\text{rank}[m - d_1 - 1, n - d_2] < \text{rank}[m - d_1, n - d_2]$, $\partial p^*_{m-d_1-1,n-d_2} < m - d_1 - d_1^*$ and $\partial q^*_{m-d_1-1,n-d_2} < n - d_2 - d_2^*$. Consequently, $p^*_{m-d_1-1,n-d_2}$, $q^*_{m-d_1-1,n-d_2}$ is also a solution of the Newton–Padé problem of order $[k, l]$, where $[k, l]$ belongs to the above defined triangle. Hence $i + j \geqslant d_1^* + d_2^*$. ∎

Note that one pair of sides of this parallelogram consists of vertical lines and that the other pair is parallel with the antidiagonal.

To get a theorem that further enlightens the structure of the table of minimal solutons, we first introduce a polynomial $s_1(z)$. We suppose that $\partial p^*_{m-d_1,n-d_2} = m - d_1$, $\partial q^*_{m-d_1,n-d_2} = n - d_2$, and $\sigma(q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2}) = m + n + 1$. This implies that we can write

$$q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2} = \omega_{0,m+n+1}(z) \cdot v(z),$$

where $v(z) \in H(E)$. Let then $z_{m+n+1+\alpha_1}$ be the first point in the sequence $\{z_{m+n+1+j}\}_{j=1}^{\infty}$ for which $v(z) = 0$. Then

$$v(z) = (z - z_{m+n+1+\alpha_1})\, v_1(z),$$

with $v_1(z) \in H(E)$. Next let $z_{m+n+1+\alpha_2}$ be the first point in the sequence $\{z_{m+n+1+j}\}_{j=\alpha_1+1}^{\infty}$ for which $v_1(z) = 0$. Continuing this process we obtain at last

$$\begin{aligned} &q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2} \\ &\quad = \omega_{0,m+n+1}(z) \cdot (z - z_{m+n+1+\alpha_1})(z - z_{m+n+1+\alpha_2}) \cdots w(z) \end{aligned}$$

where $w(z) \in H(E)$. Then $s_l(z)$ is defined by

$$s_l(z) = \frac{\prod_{j=1}^{l} (z - z_{m+n+j})}{S_l(z)},$$

with

$$\begin{aligned} S_l(z) &= 1, \qquad \text{if} \quad l < 1 + \alpha_1, \\ &= \prod_{j=1}^{t} (z - z_{m+n+1+\alpha_j}), \qquad \text{if} \quad 1 + \alpha_1 \leqslant 1 + \alpha_t \leqslant l < 1 + \alpha_t. \end{aligned}$$

Before proving the theorem we note that the points $z_{m+n+1+\alpha_j}$, $j = 1, 2,...,$ can also be characterized in another equivalent way. Therefore we have to remember that $r_{mn}(z)$ can also be treated as a rational interpolant, interpolating the function $f(z)$ in the Hermite sense in the points $\{z_i\}_{i=0}^{m+n}$ (see [1]). The points $z_{m+n+1+\alpha_j}$, $j = 1, 2,...,$ are then the interpolation points in which r_{mn} still satisfies the interpolation condition and for which

$$z_{m+n+1+\alpha_j} \notin (\{z_{m+n+1+k}\}_{k=0}^{\alpha_j-1} \backslash \{z_{m+n+1+\alpha_i}\}_{i=1}^{j-1}) \cup \{z_{\beta_i}\}_{i=1}^{l'},$$

where z_{β_i} $(i = 1, 2,..., l')$ denote the possible unattainable points of r_{mn}. Or, in words, once a point z_i is unattainable for r_{mn}, then all the interpolation points $z_j = z_i$ and $j > i$ have to be excluded. Note that in the sets defined above $\{z_{m+n+1+\alpha_i}\}_{i=1}^{j-1} = \varnothing$ if $j = 1$, and that $\{z_{\beta_i}\}_{i=1}^{l'} = \varnothing$ if r_{mn} has no unattainable points.

That this is an equivalent way of characterizing the points $z_{m+n+1+\alpha_j}$, $j = 1, 2,...,$ can be seen by using the lemma of Salzer [3, p. 487].

With the above definitions and notations we can formulate the following theorem.

THEOREM 6. *If* $\partial p^*_{m-d_1,n-d_2} = m - d_1$, $\partial q^*_{m-d_1,n-d_2} = n - d_2$, *and* $\sigma(q^*_{m-d_1,n-d_2} f - p^*_{m-d_1,n-d_2}) = m + n + 1$, *then if* $l \leqslant 2t + d_1 + d_2$,

$$p^*_{m-d_1+i,n+d_1+l-i} = s_l(z) \cdot p^*_{m-d_1,n-d_2},$$

$$q^*_{m-d_1+i,n+d_1+l-i} = s_l(z) \cdot q^*_{m-d_1,n-d_2}$$

for $i = k, k + 1,..., d_1 + d_2 + l - k$, *where* $l = 1, 2,...,$ *and* $k = l - t$, *and where* $s_l(z)$ *is the polynomial defined above.*

Proof. From Theorem 3 we know that the minimal solutions lying in the triangle with corners $[m - d_1, n - d_2]$, $[m - d_1, n + d_1]$, and $[m + d_2, n - d_2]$ are all equal to each other. The triangle is even a maximal triangle of equal minimal solutions. To prove the theorem we first show that

$p = s_l(z) \cdot p^*_{m-d_1,n-d_2}$, $q = s_l(z) \cdot q^*_{m-d_1,n-d_2}$ with $s_l(z)$ defined as above, is a solution of the Newton–Padé problem of order $[m - d_1 + i, n + d_1 + l - i]$ if $l \leqslant 2t + d_1 + d_2$.

Now, since $\partial S_l = t$ we have that

$$\partial p = m - d_1 + l - t = m - d_1 + k$$

and

$$\partial q = n - d_2 + k.$$

Under the hypothesis that $k \leqslant l + d_1 + d_2 - k$ one has $k \leqslant l + d_1 + d_2 - i$ with $i = k, k+1, \ldots, l + d_1 + d_2 - k$. For suppose for some i, $k > l + d_1 + d_2 - i$, then certainly $k > l + d_1 + d_2 - (l + d_1 + d_2 - k) = k$, which is a contradiction. The hypothesis $k \leqslant l + d_1 + d_2 - k$ is equivalent to $l \leqslant 2l - 2k + d_1 + d_2 = 2t + d_1 + d_2$. Hence, $k \leqslant l + d_1 + d_2 - k$ if and only if $l \leqslant 2t + d_1 + d_2$.

Consequently, only if this hypothesis is satisfied may we write,

$$\partial p \leqslant m - d_1 + i,$$
$$\partial q \leqslant n - d_2 + l + d_1 + d_2 - i = n + d_1 + l - i.$$

Further, because of the way in which we have constructed $s_l(z)$, we have that

$$\sigma(qf - p) \geqslant m + n + l + 1.$$

This proves that if $l \leqslant 2t + d_1 + d_2$, then p, q is a solution of the Newton–Padé problem of order $[m - d_1 + i, n + d_1 + l - i]$. Now we show this solution is the minimal solution.

To this end suppose that p, q is not the minimal solution. Then by Theorem 1, the minimal solution must have the form

$$\frac{s_l(z) \cdot d_{m-d_1,n-d_2}(z)\, p_{m-d_1,n-d_2}}{P(z)}, \frac{s_l(z) \cdot d_{m-d_1,n-d_2}(z)\, q_{m-d_1,n-d_2}}{P(z)},$$

where $P(z)$ is a polynomial and $d_{m-d_1,n-d_2}(z)$ is the greatest common divisor of $p^*_{m-d_1,n-d_2}$, $q^*_{m-d_1,n-d_2}$. Since $p_{m-d_1,n-d_2}$, $q_{m-d_1,n-d_2}$ are relatively prime, $P(z)$ must divide $s_l(z) \cdot d_{m-d_1,n-d_2}(z)$. By construction, this is impossible without violating the condition $\sigma(qf - p) \geqslant m + n + l + 1$. Hence $P(z)$ has to be a constant factor. And this implies that p, q is the minimal solution for the Newton–Padé problem of order $[m - d_1 + i, n + d_1 + l - i]$. ∎

3. Characterization Theorems for the Newton–Padé Table

The theorems of the previous section explain the structure of the minimal solution table. From these theorems we can immediatly derive some theorems enlightening the structure of the Newton–Padé table.

THEOREM 7. *If* $\partial p^*_{m-d_1,n-d_2} = m - d_1$, $\partial q^*_{m-d_1,n-d_2} = n - d_2$, *and* $\sigma(q^*_{m-d_1,n-d_2}f - p^*_{m-d_1,n-d_2}) = m + n + 1$, *then all the Newton–Padé approximants lying in the square of the Newton–Padé table determined by* $[m - d_1, n - d_2]$ *and* $[m + d_2, n + d_1]$ *are identical.*

Proof. It is clear that the assertion is true for the elements lying in the triangle determined by $[m - d_1, n - d_2]$, $[m - d_1, n + d_1]$, and $[m + d_2, n - d_2]$, since by Theorem 3 all these elements have the same minimal solution. We show that, whatever the following interpolation conditions may be, this is also the case for the elements lying in the triangle determined by $[m - d_1 + 1, n + d_1]$, $[m + d_2, n - d_2 + 1]$, and $[m + d_2, n + d_1]$. Hence we have to show that $r_{m-d_1+i,n-d_2+j} \equiv r_{m-d_1,n-d_2}$ for $1 \leqslant i, j \leqslant d_1 + d_2$ and for $m + n + 1 \leqslant m + n - d_1 - d_2 + i + j$ which is equivalent to $d_1 + d_2 + 1 \leqslant i + j$. Now, using the definition, it is clear that the Newton–Padé problem of order $[m - d_1 + i, n - d_2 + j]$ certainly is satisfied by $p = s(z) \cdot p^*_{m-d_1,n-d_2}$, $q = s(z) \cdot q^*_{m-d_1,n-d_2}$, with

$$s(z) = \prod_{i=m+n+1}^{m+n-d_1-d_2+i+j} (z - z_i).$$

The theorem is proved if we can show that $\partial p \leqslant m - d_1 + i$ and $\partial q \leqslant n - d_2 + j$. Considering the expressions for p and q we get:

$$\partial p \leqslant (m - d_1) + (i + j - d_1 - d_2) = m - 2d_1 - d_2 + i + j \leqslant m - d_1 + i,$$
$$\partial q \leqslant (n - d_2) + (i + j - d_1 - d_2) = n - d_1 - 2d_2 + i + j \leqslant n - d_2 + j,$$

which completes the proof. ∎

Using Theorem 4 and a similar argument as in the previous theorem, a slightly more general theorem can be proved.

THEOREM 8. *If* $\partial p^*_{m-d_1+i,n-d_2+j} = m - d_1$, $\partial q^*_{m-d_1+i,n-d_2+j} = n - d_2$, *and* $\sigma(q^*_{m-d_1+i,n-d_2+j}f - p^*_{m-d_1+i,n-d+_2j}) = m + n + 1$, where $i, j \geqslant 0$ *and* $i + j \leqslant d_1 + d_2$, *and if* $\operatorname{rank}[m - d_1, n - d_2] = n - d_2$, *then all the Newton–Padé approximants lying in the square determined by* $[m - d_1, n - d_2]$ *and* $[m + d_2, n + d_1]$ *are identical.*

As an immediate consequence of Theorem 6, we get a corresponding theorem enlightening the structure of the Newton–Padé table.

THEOREM 9. *Under the same hypothesis as formulated in Theorem* 6, *if* $l \leqslant 2t + d_1 + d_2$,

$$r_{m-d_1+i,n+d_1+l-i} \equiv r_{m-d_1,n-d_2}$$

for $i = k, k + 1,..., d_1 + d_2 + l - k$, *where* $l = 1, 2,...$ *and* $k = l - t$.

Before giving some examples, we first remark that the structure of the Newton–Padé table can also be pictured in a way different from the one given in Theorem 9.

THEOREM 9*. *Under the same hypothesis as formulated in Theorem* 6, *if* $\alpha_t < 2t + d_1 + d_2$, *then*

$$r_{m+d_2+t,n-d_2+l-t} = r_{m-d_1,n-d_2} = r_{m-d_1+l-t,n+d_1+t}$$

for $l = \alpha_t + 1, \alpha_t + 2,..., d_1 + d_2 + 2t$, *where* $t = 1, 2,...$.

The proof of this theorem parallels the proof of Theorem 6.

As is well known [2, p. 13], the Padé table (which is a special case of the Newton–Padé table) has a square block structure. If there are two equal elements in the Padé table there exists a square block of equal elements. However, once the block is determined, no other elements of the table will be equal to the elements of this block.

As follows from Theorem 9*, in general the Newton–Padé table will not have a square block structure. However, one may say that the starting point is still a square block (eventually of length 1). This block can have a sort of tail concentrated symmetrically along its main diagonal (see Examples 2 and 3). A second difference with the Padé table lies in the fact that further on in the table there may occur elements which are equal to the elements of the block. These elements, however, will again be concentrated along the same diagonal, and will have an analogous structure (see Example 2).

4. SOME EXAMPLES

To conclude we give two examples, illustrating the structure of the Newton–Padé table and of the table of minimal solutions.

EXAMPLE 2. *Let* $z_{0+4i} = -3$, $z_{1+4i} = 0$, $z_{2+4i} = 1$, $z_{3+4i} = 2$, $z_{12+i} = 3$ *for* $i = 0, 1, 2$, *and let* $f(-3) = \frac{1}{2}$, $f(0) = 2$, $f(1) = \frac{3}{2}$, $f(2) = \frac{4}{3}$, $f'(-3) = -\frac{1}{4}$, $f'(0) = 1$, $f'(1) = 1$, $f'(2) = -\frac{1}{9}$, $f''(-3) = 1$, $f''(0) = 2$, $f''(1) = 1$, $f''(2) = \frac{2}{27}$, $f(3) = \frac{5}{4}$, $f'(3) = 1$, $f''(3) = 1$.

Then $p_{11}^* = z + 2$, $q_{11}^* = z + 1$, and $\sigma(q_{11}^* f - p_{11}^*) = 5$.

(a) *Structure of the Newton–Padé table* (*see Table I*). In view of Theorem 7, all the Newton–Padé approximants lying in the square determinded by [1, 1] and [3, 3] are equal to $r_{11} = (z+2)/(z+1)$. Using Theorem 9* we see that:

$$\alpha_1 = 2 < d_1 + d_2 + 2t = 4,$$
$$\alpha_2 = 6 \not< d_1 + d_2 + 2t = 6,$$
$$\alpha_3 = 7 < d_1 + d_2 + 2t = 8.$$

TABLE I

Newton–Padé Table for Example 2

r_{mn}	1	2	3	4	5	6
1	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$			
2	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$			
3	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$		
4			$\frac{z+2}{z+1}$	$\frac{z+2}{z+1}$		
5						
6						$\frac{z+2}{z+1}$

Hence,

$$r_{4,l} \equiv r_{11} \equiv r_{1,4}, \qquad \text{for} \quad l = 3, 4,$$
$$r_{6,-2+l} \equiv r_{11} \equiv r_{-2+l,6}, \qquad \text{for} \quad l = 8.$$

Remark that although $r_{11}''(z)\,|_{z=0} = 2$, $z_9 \in [(\{z_{5+i}\}_{i=0}^{3} \backslash \{z_7\}) \cup \varnothing]$, viz., $z_9 = z_5$.

(b) *Structure of the table of minimal solutions* (*see Table II*). In view of Theorem 3 the triangle determined by [1, 1], [1, 3], and [3, 1] is a maximal triangle of equal minimal solutions, viz., equal to $p_{11}^* = z + 2$, $q_{11}^* = z + 1$.

Using Theorem 6 we see that

$$l \leqslant 2t + d_1 + d_2$$

TABLE II

Table of Minimal Solutions for Example 2

p^*_{mn} q^*_{mp}	1	2	3	4	5	6
1	$z+2$ $z+1$	$z+2$ $z+1$	$z+2$ $z+1$			
2	$z+2$ $z+1$	$z+2$ $z+1$	$(z+2)z$ $(z+1)z$			
3	$z+2$ $z+1$	$(z+2)z$ $(z+1)z$	$(z+2)z(z-1)$ $(z+1)z(z-1)$	$(z+2)z(z-1)$ $(z+1)z(z-1)$		
4			$(z+2)z(z-1)$ $(z+1)z(z-1)$	$(z+2)z(z-1)(z+3)$ $(z+1)z(z-1)(z+3)$		
5						
6						$(z+2)z^2(z-1)^2(z+3)$ $(z+1)z^2(z-1)^2(z+3)$

for $l = 1, 2, 3, 4, 8$. Consequently:

$$l = 1, \quad \begin{cases} p^*_{1+i,\lambda-i+1} = (z+2)\,z \\ q^*_{1+i,\lambda-i+1} = (z+1)\,z \end{cases} \quad \text{for } i = 1, 2 \text{ and } \lambda = 3,$$

$$l = 2, \quad \begin{cases} p^*_{1-i,\lambda-i+1} = (z+2)\,z(z-1) \\ q^*_{1-i,\lambda-i+1} = (z+1)\,z(z-1) \end{cases} \quad \text{for } i = 2 \text{ and } \lambda = 4,$$

$$l = 3, \quad \begin{cases} p^*_{1+i,\lambda-i+1} = (z+2)\,z(z-1) \\ q^*_{1+i,\lambda-i+1} = (z+1)\,z(z-1) \end{cases} \quad \text{for } i = 2, 3 \text{ and } \lambda = 5,$$

$$l = 4, \quad \begin{cases} p^*_{1+i,\lambda-i+1} = (z+2)\,z(z-1)(z+3) \\ q^*_{1+i,\lambda-i+1} = (z+1)\,z(z-1)(z+3) \end{cases} \quad \text{for } i = 3 \text{ and } \lambda = 6,$$

$$l = 8, \quad \begin{cases} p^*_{1+i,\lambda-i+1} = (z+2)\,z^2(z-1)^2\,(z+3) \\ q^*_{1+i,\lambda-i+1} = (z+1)\,z^2(z-1)^2\,(z+3) \end{cases} \quad \text{for } i = 5 \text{ and } \lambda = 10.$$

EXAMPLE 3. *Let* $z_{j+6i} = -3 + j$ for $j = 0, 1,\ldots, 5$ and $i = 0, 1, 2$. And let $f(-3) = 1, f(-2) = 3, f(-1) = 3, f(0) = 10, f(1) = 5, f(2) = 6$, $f'(-3+j) = 1$ for $j = 0, 1,\ldots, 3$, $f'(1) = 3$, $f'(2) = 1$, $f''(-3+j) = 0$ for $j = 0, 1,\ldots, 4$ and $f''(2) = 5$.

Then $p^*_{32} = (z+4)(z+2)\,z$, $q^*_{32} = (z+2)\,z$, and $\sigma(q^*_{32}f - p^*_{32}) = 7$.

(a) *Structure of the Newton–Padé table* (*see Table III*). In view of Theorem 7 we conclude that all the Newton–Padé approximants lying in the square determined by [3, 2] and [4, 3] reduce to $z + 4$.

TABLE III

Newton–Padé Table for Example 3

r_{mn}	2	3	4	5	6	7
3	$z+4$	$z+4$				
4	$z+4$	$z+4$	$z+4$			
5		$z+4$	$z+4$			
6				$z+4$	$z+4$	
7				$z+4$	$z+4$	$z+4$
8					$z+4$	$z+4$

TABLE IV

Table of Minimal Solutions for Example 3

p^*_{mn} / q^*_{mn}	2	3	4	5	6	7
3	$(z+4)(z+2)z$ $(z+2)z$	$(z+4)(z+2)z$ $(z+2)z$				
4	$(z+4)(z+2)z$ $(z+2)z$	$(z+4)(z+2)^2z$ $(z+2)^2z$	$(z+4)(z+2)^2z$ $(z+2)^2z$			
5		$(z+4)(z+2)^2z$ $(z+2)^2z$	$(z+4)(z+2)^2z^2$ $(z+2)^2z^2$			
6				$(z+4)(z+2)^2z^2(z-1)$ $(z+2)^2z^2(z-1)$	$(z+4)(z+2)^2z^2(z-1)$ $(z+2)^2z^2(z-1)$	
7				$(z+4)(z+2)^2z^2(z-1)$ $(z+2)^2z^2(z-1)$	$(z+4)(z+2)^3z^3(z-1)$ $(z+2)^3z^2(z-1)$	$(z+4)(z+2)^3z^2(z-1)$ $(z+2)^3z^2(z-1)$
8					$(z+4)(z+2)^3z^2(z-1)$ $(z+2)^3z^2(z-1)$	$(z+4)(z+2)^3z^3(z-1)$ $(z+2)^3z^3(z-1)$

Using Theorem 9* we see that:

$$\alpha_1 = 1 < d_1 + d_2 + 2t = 3,$$
$$\alpha_2 = 4 < d_1 + d_2 + 2t = 5,$$
$$\alpha_3 = 5 < d_1 + d_2 + 2t = 7,$$
$$\alpha_4 = 7 < d_1 + d_2 + 2t = 9.$$

Hence,

$$r_{4+t,2-t+l} \equiv r_{32} \equiv r_{3-t+l,3+t}$$

for $t = 1, 2, 3, 4$ and $l = \alpha_t + 1,..., d_1 + d_2 + 2t$.

(b) *Structure of the table of minimal solutions* (*see Table IV*). In view of Theorem 3, the elements of order [3, 2], [3, 3], and [4, 2] have the same minimal solution.

Using Theorem 6 we see that

$$l \leqslant 2t + d_1 + d_2$$

for $l = 1, 2,..., 9$ with the exception of $l = 4$. The further results are displayed in Table IV.

References

1. G. Claessens, On the Newton–Padé approximation problem, *J. Approximation Theory* **22** (1978), 150–160.
2. W. B. Gragg, The Padé table and its relation to certain algorithms of numerical analysis, *SIAM Rev.* **14** (1972), 1–62.
3. H. E. Salzer, Note on osculatory rational interpolation, *Math. Comp.* **16** (1962), 486–491.